Circular edge-colourings of cubic graphs with girth at least six

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Cubic graphs

- 3-edge-colourable ... $\chi' = 3$
- non-3-edge-colourable ... $\chi' = 4$
Cubic graphs

- 3-edge-colourable \( \chi' = 3 \)
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Snarks – non-3-edge-colourable bridgeless cubic graphs
Cubic graphs

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Snarks – non-3-edge-colourable bridgeless cubic graphs

Snarks are the smallest counterexamples to many open conjectures:

- the Cycle Double Cover Conjecture
- Tutte’s 5-flow conjecture
- the Fulkerson conjecture
Girth Conjecture [Jaeger, Swart 1980]

There are no snarks with large girth.
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Theorem [Kochol 1996]
There are snarks with arbitrarily large girth.
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Theorem [Kochol 1996]
There are snarks with arbitrarily large girth.

is it true that snarks with large girth are close to being 3-edge-colourable?
Circular edge-colourings of snarks with girth at least 6

$(p, q)$-edge-colouring

- colours from $\{0, 1, 2, \ldots, p - 1\}$
- any two adjacent edges receive colours $a$ and $b$ such that

$$q \leq |a - b| \leq p - q$$

$(p, q)$-edge-colouring

$G$ is cubic 3-edge-colourable $\Rightarrow \chi'_{\text{c}}(G) = 3$
(\(p, q\))-edge-colouring

- colours from \(\{0, 1, 2, \ldots, p - 1\}\)
- any two adjacent edges receive colours \(a\) and \(b\) such that

\[q \leq |a - b| \leq p - q\]

circular chromatic index \(\chi'_c\)

- the infimum of the ratios \(p/q\) such that \(G\) has a \((p, q)\)-colouring
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\((p, q)\)-edge-colouring

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circular chromatic index \(\chi'_c\)

- the infimum of the ratios \(p/q\) such that \(G\) has a \((p, q)\)-colouring
- the infimum is the minimum
- \(\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G)\)
- \(G\) is cubic 3-edge-colourable \(\Rightarrow \chi'_c(G) = 3\)
(11, 3)-colouring of the Petersen graph
circular chromatic index $\chi'_c$

Circular chromatic index $\chi'_c$ of snarks

- $\chi'_c(G) \leq 11/3$ [Afshani et. al. 2005]
Circular edge-colourings of snarks with girth at least 6

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- $\chi'_c(G) \leq 11/3$ [Afshani et. al. 2005]
- $\forall \varepsilon > 0 \exists g :$ every snark $G$ of girth at least $g$ has $\chi'_c(G) \leq 3 + \varepsilon$
  [Kaiser, Král’, Škrekovski, 2007]
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- $\chi'_c$ is known for Isaacs snarks, Goldberg snarks, generalised Blanuša snarks
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- $\chi'_c(Pg) = 11/3$
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- the Petersen graph is the only known cubic bridgeless graph with $\chi'_c(G) > 7/2$
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Conjecture

$G$ – cubic bridgeless; $G \neq Pg$ then $\chi'_c(G) < 11/3$ (maybe $\chi'_c(G) \leq 7/2$)
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circular chromatic index $\chi'_c$

Problem [Kaiser, Král’, Škrekovski, 2007]

Determine the smallest girth $g_0$ such that every cubic graph $G$ with girth at least $g_0$ has $\chi'_c(G) \leq 7/2$.

Theorem [Král’, Mácajová, M., and Sereni, 2009+]

$G$ – cubic bridgeless

$\chi'_c(G) \leq 7/2$

$g_0 = 6$
Problem [Kaiser, Král’, Škrekovski, 2007]

Determine the smallest girth $g_0$ such that every cubic graph $G$ with girth at least $g_0$ has $\chi_c'(G) \leq 7/2$.

- $g_0 \leq 14$
circular chromatic index $\chi'_c$

Problem [Kaiser, Král’, Škrekovski, 2007]
Determine the smallest girth $g_0$ such that every cubic graph $G$ with girth at least $g_0$ has $\chi'_c(G) \leq 7/2$.

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Theorem [Král’, Máčajová, M., and Sereni, 2009+]
$G$ – cubic bridgeless
$g(G) \geq 6 \Rightarrow \chi'_c(G) \leq 7/2$
Circular edge-colourings of snarks with girth at least 6

circular chromatic index $\chi'_c$

**Problem [Kaiser, Král’, Škrekovski, 2007]**

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**Theorem [Král’, Máčajová, M., and Sereni, 2009+]**

$G$ – cubic bridgeless

$g(G) \geq 6 \Rightarrow \chi'_c(G) \leq 7/2$

- $g_0 = 6$
Proof – sketch

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

Let $G$ be a bridgeless cubic graph with girth at least 6. Then $\chi'_c(G) \leq 7/2$.
Proof – sketch

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

\[ G \text{ – bridgeless cubic} \]
\[ g(G) \geq 6 \Rightarrow \chi'_c(G) \leq 7/2 \]

- bridgeless cubic graph contains a 1-factor (Petersen theorem)
Proof – sketch

Theorem [Král', Máčajová, M., and Sereni, 2009+]

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- \( F \) ... 1-factor  \( C \) ... 2-factor
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- we colour
  - \( F \) by colours 0, 1
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- bridgeless cubic graph contains a 1-factor (Petersen theorem)
- \( F \) ... 1-factor \( C \) ... 2-factor
- we colour
  - \( F \) by colours 0, 1
  - \( C \) by colours 2, 3, 4, 5, 6
    - even cycles 3, 5, 3, 5, ..., 3, 5
    - odd cycles 2, 4, 6, 3, 5, 3, 5, ..., 3, 5
Proof – sketch

Circular edge-colourings of snarks with girth at least 6

Ján Mazák (Bratislava)
Proof – sketch

... system of open trails with some additional properties
Proof – sketch

... system of open trails with some additional properties
Generalisations of the result

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

$G$ – subcubic

$g(G) \geq 6 \Rightarrow \chi'_c(G) \leq 7/2$
Generalisations of the result

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

\( G \) – subcubic
\[ g(G) \geq 6 \Rightarrow \chi'_c(G) \leq \frac{7}{2} \]

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

\( G \) – cubic bridgeless
\( G \) contains a 2-factor with no 5-cycle \( \Rightarrow \chi'_c(G) \leq \frac{7}{2} \)
Generalisations of the result

**Theorem** [Král’, Máčajová, M., and Sereni, 2009+]

*G* – subcubic  
\( g(G) \geq 6 \Rightarrow \chi'_c(G) \leq 7/2 \)

**Theorem** [Král’, Máčajová, M., and Sereni, 2009+]

*G* – cubic bridgeless  
*G* contains a 2-factor with no 5-cycle  
\( \Rightarrow \chi'_c(G) \leq 7/2 \)

- problem with 5-cycles
Generalisations of the result

Theorem [Král’, Máčajová, M., and Sereni, 2009+]

\[ G \text{ – subcubic} \]
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Theorem [Král’, Máčajová, M., and Sereni, 2009+]

\[ G \text{ – cubic bridgeless} \]
\[ G \text{ contains a 2-factor with no 5-cycle} \Rightarrow \chi'_c(G) \leq 7/2 \]

- problem with 5-cycles

Problem

Is the Petersen graph the only cubic bridgeless graph with \( \chi'_c(G) > 7/2 \)?
Thank you!