The Strong Perfect Graph Conjecture

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Question

For which graphs $\chi(G) = \omega(G)$?
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### Definition

A graph $G$ is called **perfect** if $\chi(H) = \omega(H)$ for each induced subgraph $H$ of $G$. 

The complement of a perfect graph is perfect.

Examples of perfect graphs:
- cliques
- bipartite graphs (and their complements)
- line graphs of bipartite graphs (and their complements)
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- line graphs of bipartite graphs (and their complements)
Definition

- an *odd hole* – an odd cycle with at least four vertices
- an *odd anti-hole* – the complement of an odd hole

A graph is a Berge graph if it contains neither an odd hole nor an odd anti-hole as an induced subgraph.

The Strong Perfect Graph Conjecture (SPGC)

A graph is perfect if and only if it is a Berge graph.
Berge Graphs

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Triangulated graphs

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- **Clique cutset** – a cutset formed by a clique.

Theorem (Dirac)

A triangulated graph is either a clique or has a clique cutset.

Every triangulated graph is perfect.
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Every triangulated graph is perfect.
The join $G + H$ of two graphs $G$ and $H$: each vertex of $G$ is joined to each vertex of $H$. The join $G + H$ is perfect if and only if both $G$ and $H$ are perfect.
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- each vertex of $G$ is joined to each vertex of $H$.

The join $G + H$ is perfect $\iff$ both $G$ and $H$ are perfect.

Definition
Let $A, B$ be a partition of $V(G)$;
- $A_1, A_2$: disjoint subsets of $A$,
- $B_1, B_2$: disjoint subsets of $B$.
If each vertex of $A_i$ is joined to each vertex of $B_i$ and there are no other edges between $A$ and $B$ then $G$ admits a 2-join.

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Perfect Graphs
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The join $G + H$ is perfect $\iff$ both $G$ and $H$ are perfect.

Let $A$, $B$ be a partition of $V(G)$; $A_1$, $A_2$: disjoint subsets of $A$, $B_1$, $B_2$: disjoint subsets of $B$.

If each vertex of $A_i$ is joined to each vertex of $B_j$ and there are no other edges between $A$ and $B$ then $G$ admits a 2-join.

The 2-join of two perfect graphs is perfect.
Definition

A *minimal imperfect graph* – a graph that is not perfect but all its induced subgraphs are.
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A counterexample to SPGC: a Berge graph that is not perfect. A smallest counterexample must be minimal imperfect.
Smallest counterexample and decompositions

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A smallest counterexample does not admit a join.
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A counterexample to SPGC: a Berge graph that is not perfect. A smallest counterexample must be minimal imperfect.

A smallest counterexample does not admit a join.

**The Star Cutset Lemma (Chvátal, 1985)**

No minimal imperfect graph has a star cutset.

This is only a *decomposition theorem*, there is no corresponding perfection-preserving composition.
Definition

A *skew partition* of $G$ – a partition $(A, B, C, D)$ of $V(G)$ such that all vertices of $A$ are joined to all of $B$ and there are no edges between $C$ and $D$. The set $A \cup B$ forms a *skew cutset*.
**Definition**

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**Conjecture (Chvátal 1985)**

No minimal imperfect graph admits a skew partition.
Theorem (Cornuéjols et. al. 2001)

A square-free Berge graph $G$ satisfies at least one of the following statements:

1. $G$ has a star cutset,
2. $G$ admits a 2-join,
3. $G$ is bipartite or is the line graph of a bipartite graph.
Theorem (Cornuéjols et. al. 2001)

A square-free Berge graph $G$ satisfies at least one of the following statements:

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Conjecture (Cornuéjols et. al.)

A Berge graph $G$ satisfies at least one of the following statements:

1. $G$ or $\overline{G}$ admits a 2-join,
2. $G$ or $\overline{G}$ has a skew partition,
3. $G$ or $\overline{G}$ is bipartite or is the line graph of a bipartite graph.
Conjecture (Seymour et. al.)

A Berge graph $G$ satisfies at least one of the following statements:

1. $G$ belongs to one of the basic classes, that is, either
   - $G$ or $\overline{G}$ is bipartite, or
   - $G$ or $\overline{G}$ is a line graph of a bipartite graph, or
   - $G$ is a double-split graph;

2. $G$ or $\overline{G}$ admits a 2-join;

3. $G$ has a balanced skew partition;